GENERATION OF VORTEX FLOWS IN AN INCOMPRESSIBLE CONDUCTING VISCOUS FLUID BY AN ALTERNATING ELECTROMAGNETIC FIELD

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§1. Variable electromagnetic fields are used in many magnetohydrodynamic processes, e.g., to contain and stabilize a plasma [1], to mix liquid metals [2], to control casting [3], etc. In most cases the Lorentz forces $(1/c)[\mathbf{j}\times\mathbf{H}]$ in a conducting medium are nonpotential and, consequently, cannot be balanced by a pressure gradient; they thus lead to the generation of vortex flow of the conducting medium. In addition, the nonpotential electromagnetic forces may turn out to be a convenient means for producing "standard" vortex flows for research purposes, since interest in these flows is by no means exhausted [4].

As a consequence of the complex nature of the distribution of Lorentz forces, the induced flow can be very unusual; this increases the interest in investigating vortex flows in variable electromagnetic fields.

Since the magnetohydrodynamic equations are nonlinear, these problems can generally be investigated only by modern numerical methods. By making some simplifying assumptions leading to a linearization of the equations, we obtain accurate solutions of two problems (outer and inner) with a spherical boundary between a conducting fluid and nonconducting space.

In the outer problem a conducting fluid fills all of space outside a solid nonconducting sphere of radius r_0. An alternating electromagnetic field of frequency ω is produced by an alternating current localized in the immediate vicinity of the center of the sphere. Consequently, the system of currents is replaced by an alternating magnetic moment as a first approximation. We consider the case of a magnetic moment of fixed direction varying only in magnitude, i.e., $\mathbf{m}=m_0e^{i\omega t}\mathbf{e}_z$ (Fig. 1a).

In the inner problem a conducting fluid fills a spherical cavity; the cavity and fluid are in an external alternating magnetic field $\mathbf{H}_0e^{i\omega t}$, $\mathbf{H}_0=H_0\mathbf{e}_z=H_0(\cos\theta\,\mathbf{e}_r-\sin\theta\,\mathbf{e}_\theta)$ (Fig. 1b).

Henceforth we shall speak of the problems shown schematically in Fig. 1 as problems *a* and b.

The solutions are obtained under the following assumptions:

1. The Stokes approximation is valid to describe the flow;

2. the magnetic Reynolds number is small; i.e.,

$$\mathrm{Re}_m = 4\pi\sigma v_0 r_0/c^2 \ll 1 \tag{1.1}$$

where v_0 is a characteristic velocity of the generated flow; ν and σ are the kinematic viscosity and the conductivity of the fluid;

3. the behavior of the system is investigated after the periodic regime has been established; the process of reaching this regime is not examined;

4. the frequency ω satisfies the quasistationary condition; i.e., $(\omega/c)r_0\ll 1$;

5. the magnetic permeability μ and the dielectric permittivity ε are everywhere equal to unity.

Actually, condition (1.1) follows from assumption 1, since for all conducting fluids, including electrolytes and liquid metals, $\nu_m = c^2/4\pi\sigma \gg \nu$. Assumptions 5 are not necessary for the solution and are made in order to simplify the final formulas.

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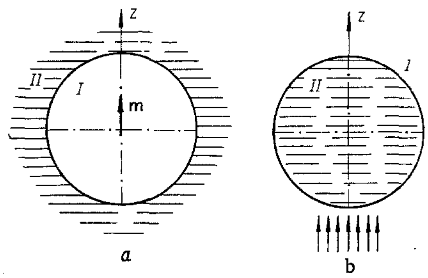

Fig. 1

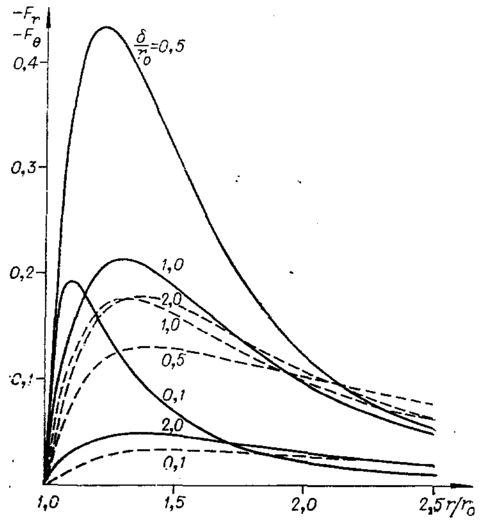

Fig. 2

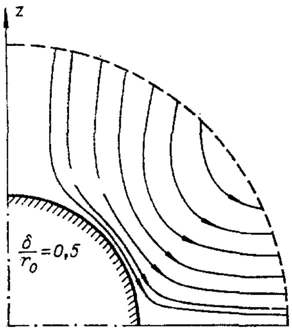

Fig. 3

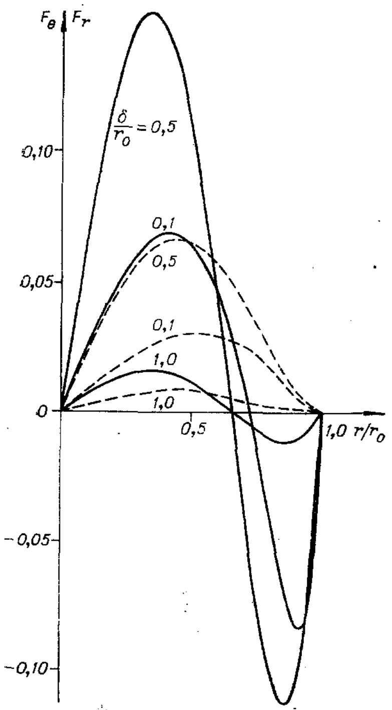

Fig. 4

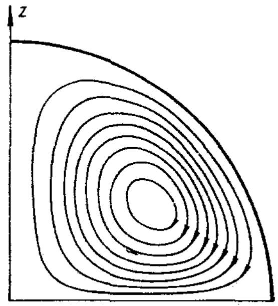

Fig. 5

The process under investigation is described by the equations of magnetohydrodynamics. In view of assumption (1.1) the motion of the fluid does not affect the variation of the electric and magnetic fields. The electrodynamics problem and the problem of determining the flow generated by the Lorentz forces are thus separated. An analogous problem of the flow of a fluid in an infinitely long cylinder is solved in [5] for an external variable magnetic field at right angles to the axis of the cylinder.

§2. It is convenient to calculate the **E** and **H** fields in the problems under consideration in terms of the vector potential **A**:

$$\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t; \quad \mathbf{H} = \operatorname{rot}\mathbf{A}.$$

In spherical coordinates (r, θ, α) associated with the boundary of the media the vector **A** has only one component $\mathbf{A} = A(r, \theta, y)\mathbf{e}_\alpha$; because of axial symmetry $\partial/\partial\alpha \equiv 0$. The vector **A** is determined by the equations

$$\Delta \mathbf{A}_1 = 0; \tag{2.1}$$

$$\partial\mathbf{A}_2/\partial t = (c^2/4\pi\sigma)\Delta\mathbf{A}_2 \tag{2.2}$$

and the boundary conditions

$$A_1|_{r=r_0} = A_2|_{r=r_0}; \tag{2.3}$$

$$\left.\frac{\partial A_1}{\partial r}\right|_{r=r_0} = \left.\frac{\partial A_2}{\partial r}\right|_{r=r_0}; \tag{2.4}$$

$$A_2|_{r\to\infty} \neq \infty; \tag{2.5a}$$

$$A_1|_{r\to\infty} \to (r/2)H_0 e^{i\omega t} \tag{2.5b}$$

[The subscripts 1 and 2 refer, respectively, to the nonconducting and conducting regions (Fig. 1). The numbers of the equations which apply to only one of the two problems under consideration are followed by *a* or b.]

In addition to conditions (2.3)-(2.5), the solution of problem *a* for $r = 0$ must have the singularity $(m_0/r^2)\cdot\sin\theta e^{iwt}$ due to the magnetic dipole m; the solution of problem b must be bounded.

The periodic solution of Eqs. (2.1) and (2.2) which satisfies the conditions enumerated has the form

$$A_1 = m_0\left(C_1 r + \frac{1}{r^2}\right)\sin\theta e^{i\omega t}, \quad A_2 = C_2\frac{3m_0}{r_0^{3/2}}\frac{1}{\sqrt{r}}H_{3/2}^{(2)}(kr)\sin\theta e^{i\omega t};$$

$$C_1 = \frac{1}{r_0^3}\frac{(kr_0)H_{1/2}^{(2)}(kr_0)}{3H_{3/2}^{(2)}(kr_0) - (kr_0)H_{1/2}^{(2)}(kr_0)}, \quad C_2 = \frac{1}{3H_{3/2}^{(2)}(kr_0) - (kr_0)H_{1/2}^{(2)}(kr_0)}$$

for problem *a*;

$$A_1 = \left(\frac{m_1}{r^3} + \frac{H_0}{2}\right)r\sin\theta e^{i\omega t}, \quad A_2 = D_2\frac{J_{3/2}(kr)}{\sqrt{r}}\sin\theta e^{i\omega t},$$

$$m_1 = -\frac{H_0 r_0^3}{2}\left(1 - \frac{3}{k^2 r_0^2} + \frac{3}{kr_0}\operatorname{ctg} kr_0\right), \quad D_2 = 3r_0\sqrt{\frac{\pi}{8k}}\frac{1}{\sin kr_0}H_0$$

for problem b, where $k = (1-i)/\delta$, and $\delta = c/\sqrt{2\pi\sigma\omega}$ is the thickness of the skin layer; the constant m_1 is the amplitude of the magnetic moment acquired by the conducting sphere in the field $H_0 e^{i\omega t}$; the $H_\lambda^{(2)}(x)$ are Hankel functions of the second kind [6] of order λ; and $J_{3/2}(x)$ is a Bessel function of order 3/2.

§3. The flow of an incompressible conducting fluid is described by the hydrodynamic equations

$$\operatorname{div}\mathbf{v} = 0; \tag{3.1}$$

$$\operatorname{rot}\mathbf{v} = \mathbf{w}; \tag{3.2}$$

$$\partial\mathbf{w}/\partial t + \nu\operatorname{rot}\operatorname{rot}\mathbf{w} = (1/\rho c)\operatorname{rot}[\mathbf{j}\times\mathbf{H}]. \tag{3.3}$$

Since the Stokes approximation is used, the nonlinear term rot $[\mathbf{v}\times\mathbf{w}]$ in the last equation is dropped. The Lorentz force $(1/c)[\mathbf{j}\times\mathbf{H}] = (\sigma/c)[\mathbf{E}_2\times\mathbf{H}_2]$ on the right-hand side of (3.3) is calculated from the solution of the electrodynamic part of the problem and can be written in the form

$$(1/c)[\mathbf{j}\times\mathbf{H}] = (\sigma/2c)\left[\left(\mathcal{E}_2\overset{*}{\mathcal{H}}_{2r}\mathbf{e}_\theta - \mathcal{E}_2\overset{*}{\mathcal{H}}_{2\theta}\mathbf{e}_r\right) + \left(\mathcal{E}_2\mathcal{H}_{2r}\mathbf{e}_\theta - \mathcal{E}_2\mathcal{H}_{2\theta}\mathbf{e}_r\right)e^{2i\omega t}\right], \tag{3.4}$$

where $\mathcal{E}_2$, and $\mathcal{H}_2$ are complex amplitudes in the expressions $\mathbf{E}_2(r, \theta, t) = \mathcal{E}_2(r, \theta)e^{i\omega t}$ and $\mathbf{H}_2(r, \theta, t) = \mathcal{H}_2(r, \theta)e^{i\omega t}$. It is clear from (3.4) that the force field consists of a steady part and an oscillating part with a double frequency. As a result the fluid flow also has similar components.

The vorticity of the flow $\mathbf{w}$ has a nonzero α component; i.e., $\mathbf{w}=w(r, \theta, t)\mathbf{e}_\alpha$. The following boundary conditions are known for $w(r, \theta, t)$:

$$w|_{r=\infty} \neq \infty; \tag{3.5a}$$

$$w|_{r=0} \neq \infty. \tag{3.5b}$$

For $r=r_0$ the boundary conditions for the vorticity are unknown. The solution of Eq. (3.3) is written as a sum:

$$w(r, \theta, t) = (\Phi_0(r)/r)\sin 2\theta + (\Phi_1(r)/\sqrt{r})\sin 2\theta e^{2i\omega t},$$

where the first term describes the steady part of the flow and the second, the oscillating part.

The functions $\Phi_0(r)$ for problems a and b which satisfy Eqs. (3.5a, b) have the form

$$\Phi_0(r) = C_0 r^{-2} + B_1\left[\left(\frac{3\delta}{4r^2} + \frac{1}{4r} - \frac{1}{6\delta} + \frac{r}{6\delta^2} - \frac{r^2}{3\delta^3}\right)e^{-\frac{2r}{\delta}} + \frac{2r^3}{3\delta^4}\int_r^\infty x^{-1}e^{-\frac{2x}{\delta}}dx\right]$$

for problem a;

$$\Phi_0(r) = D_0 r^3 + B_2\left[\left(\frac{3\delta}{4r^2} - \frac{r^2}{3\delta^3}\right)\left(\sin\frac{2r}{\delta} - \operatorname{sh}\frac{2r}{\delta}\right) + \frac{1}{4r}\left(\operatorname{ch}\frac{2r}{\delta} - \cos\frac{2r}{\delta}\right) + \right.$$

$$\left. + \frac{r}{6\delta^2}\left(\operatorname{ch}\frac{2r}{\delta} + \cos\frac{2r}{\delta}\right) + \frac{1}{6\delta}\left(\sin\frac{2r}{\delta} + \operatorname{sh}\frac{2r}{\delta}\right) + \frac{2r^3}{3\delta^4}\int_0^r x^{-1}\left(\cos\frac{2x}{\delta} - \operatorname{ch}\frac{2x}{\delta}\right)dx\right]$$

for problem b, where $B_1 = \frac{9}{20\pi^2}\frac{m_0^2}{r_0^3}\frac{1}{\rho\nu\delta}|C_2|^2$, $B_2 = \frac{1}{20\sqrt{2}\pi^2}\frac{1}{\rho\nu\delta}|D_2|^2$; C_0 and D_0 are arbitrary constants which will be found in determining the velocity field.

We present an expression for $\Phi_1(r)$ which is valid for problem a in the limit of a strong skin-effect; i.e., for $\delta \ll r_0$,

$$\Phi_1(r) = \alpha e^{-2(1+i)\frac{r-r_0}{\delta_1}} + \alpha_1 e^{-2(1+i)\frac{r-r_0}{\delta}},$$

$$\alpha_1 = i\left(\frac{3}{4\pi}\right)^2\frac{\sqrt{r_0}}{\rho\nu}\left(\frac{\delta_1}{\delta}\right)^2\left(\frac{m_0}{r_0^3}\right)^2 e^{-2(1+i)\frac{r_0}{\delta}}C_2^2,\ \delta_1 = 2\sqrt{\frac{\nu}{\omega}},$$

α is an undetermined constant; δ_1 is the viscous skin layer, and because $\nu \ll \nu_m = c^2/4\pi\sigma$, $\delta_1 \ll \delta$.

§4. The velocity field is determined from Eqs. (3.1) and (3.2) by using the vorticities $\mathbf{w}_0 = (\Phi_0(r)/r)\cdot\sin 2\theta \mathbf{e}_\alpha$, $\mathbf{w}_1 = (\Phi_1(r)/\sqrt{r})\sin 2\theta e^{2i\omega t}\mathbf{e}_\alpha$. Equation (3.1) is satisfied identically by the introduction of the vector potentials $\Psi_0(r, \theta)\mathbf{e}_\alpha$ and $\Psi_1(r, \theta)e^{2i\omega t}\mathbf{e}_\alpha$, respectively, for the velocities of the steady and oscillating flows

$$\mathbf{v}^0 = \operatorname{rot}[\Psi_0(r, \theta)\mathbf{e}_\alpha],\quad \mathbf{v}^1 = \operatorname{rot}[\Psi_1(r, \theta)\mathbf{e}_\alpha]e^{2i\omega t}. \tag{4.1}$$

The solutions for Ψ_0, Ψ_1 are constructed by the separation of variables

$$\Psi_0(r, \theta) = \psi_0(r)\sin 2\theta;\quad \Psi_1(r, \theta) = \psi_1(r)\sin 2\theta, \tag{4.2}$$

where according to (3.2) the functions $\psi_0(r)$ and $\psi_1(r)$ must satisfy the equations

$$(d^2/dr^2)(r\psi_0) - 6\psi_0/r = -\Phi_0(r); \tag{4.3}$$

$$(d^2/dr^2)(r\psi_1) - 6\psi_1/r = -\sqrt{r}\Phi_1(r) \tag{4.4}$$

and the boundary conditions

$$\psi_i(r_0) \neq \infty,\ (d/dr)(r\psi_i)|_{r=r_0} = 0\quad (i = 0,1),$$

$\psi_i(\infty) = 0$ for problem a, and $\psi_i(0) \neq \infty$ for problem b. Three conditions are imposed on each second-order equation (4.3) and (4.4). A solution is possible since each function $\Phi_0(r)$ and $\Phi_1(r)$ contains one arbitrary constant.

According to (4.1) and (4.2) the components of the velocity $\mathbf{v}^0$ are expressed in terms of ψ_0 by the relations

$$v_r^0 = (2\psi_0/r)(3\cos^2\theta - 1),\ v_\theta^0 = -(1/r)(d/dr)(r\psi_0)\sin 2\theta \tag{4.5}$$

with similar expressions for v_r^1 and v_θ^1.

From the solution of Eq. (4.3) the following expressions are obtained for the quantities appearing in Eqs. (4.5) for the velocity field $\mathbf{v}^0$:

$$\frac{1}{r}\psi_0(r) = V_0 F_r^0(r),\ \frac{1}{r}\frac{d}{dr}(r\psi_0) = V_0 F_\theta^0(r). \tag{4.6}$$

Since $F_\theta^0 = (1/r)d/dr\ (r^2 F_r^0)$, we present only the expressions for F_r^0. For problem *a*

$$F_r^0 = k_1 \frac{r_0}{r}\left\{7\chi(r_0)\frac{r_0}{r}\left(1-\frac{r_0^2}{r^2}\right)+2\left[\chi(r_0)\frac{r_0^3}{r^3}-\chi(r)\right]+15\left[\frac{r_0\delta^2}{r^3}\left(2+\frac{\delta}{r_0}\right)-\frac{\delta^2}{r^2}\left(2+\frac{\delta}{r}\right)e^{-\frac{2(r-r_0)}{\delta}}\right]\right\};$$

$$\chi(r) = \left(2\frac{r^2}{\delta^2}-4\frac{r^3}{\delta^3}-2\frac{r}{\delta}+3-6\frac{\delta}{r}\right)e^{-\frac{2(r-r_0)}{\delta}}+8\frac{r^4}{\delta^4}\int\limits_{r/r_0}^{\infty} x^{-1}e^{-\frac{2r_0(x-1)}{\delta}}dx,$$

$$k_1 = \left[9\frac{\delta^2}{r_0^2}\left(1+\frac{\delta}{r_0}\right)^2+\left(2+3\frac{\delta}{r_0}\right)^2\right]^{-1};$$

the dimensional constant V_0 representing the velocity scale has the value

$$V_0 = \frac{3}{140}\frac{r_0}{8\pi\rho\nu}\left(\frac{m_0}{r_0^3}\right)^2.$$

Here the parameter m_0/r_0^3 characterizes the scale of the applied magnetic field.

The dimensionless functions F_r^0 and F_θ^0 which determine the radial dependence of the components v_r^0 and v_θ^0 of the velocity of steady flow in Eqs. (4.5) and (4.6) are shown in Fig. 2 for various values of δ/r_0, with open curves for F_r^0.

If $\delta \ll r_0$, the formulas are simplified. In this case

$$F_r^0 = \frac{105}{4}\frac{\delta^2}{r_0^2}\frac{1-(r/r_0)^2}{(r/r_0)^4}; \tag{4.7a}$$

$$F_\theta^0 = \frac{105}{2}\frac{\delta^2}{r_0^2}\left(\frac{r_0}{r}\right)^4\left[1-\frac{r}{r_0}e^{-\frac{2(r-r_0)}{\delta}}\right]. \tag{4.8a}$$

It is clear from (4.8*a*) that the θ component of the velocity very rapidly (within a distance of the order of δ from the boundary) reaches its maximum value and then falls to zero as $(r_0/r)^4$. As can be seen from (4.7*a*) the radial component reaches its maximum only at $r=\sqrt{2}r_0$. The formulas presented show that for small δ the velocity of steady flow is proportional to δ^2. Figure 2 shows that the maximum values of the r and θ components of velocity are reached at $\delta/r_0 \approx 1.0$ and $\delta/r_0 \approx 0.5$, respectively. The velocities decrease for a further increase in δ.

A pictorial representation of the nature of the steady flow can be obtained from Fig. 3 which shows the flow lines [lines of constant values of $r\psi_0(r)\sin 2\theta \sin\theta$] for $\delta/r_0 = 0.5$. Since $v_\alpha \equiv 0$, the flow lines are plane curves in the planes α = const. Figure 3 shows the flow lines in the upper hemisphere; the flow in the lower hemisphere is symmetric with respect to the plane $z=0$ (or $\theta = \pi/2$) and is not shown.

For problem b the expression for F_r^0 has the form

$$F_r^0 = k_2\frac{r}{\delta}\left\{\left(1-\frac{r^2}{r_0^2}\right)\left[\chi_1(r_0)-\frac{5}{2}\chi_2(r_0)\right]+\chi_1(r)+\chi_2(r)-\chi_1(r_0)-\chi_2(r_0)\right\},$$

$$\chi_1(r) = \left(\frac{r}{3\delta}-\frac{\delta}{6r}+\frac{\delta^3}{2r^3}\right)\sin\frac{2r}{\delta}-\left(\frac{1}{6}-\frac{\delta^2}{4r^2}\right)\cos\frac{2r}{\delta}-$$

$$-\left(\frac{\delta}{6r}+\frac{r}{3\delta}+\frac{\delta^3}{2r^3}\right)\operatorname{sh}\frac{2r}{\delta}-\left(\frac{\delta^2}{4r^2}+\frac{1}{6}\right)\operatorname{ch}\frac{2r}{\delta}-\frac{2r^2}{3\delta^2}\int\limits_0^{r/\delta}x^{-1}(\cos 2x-\operatorname{ch}2x)\,dx,$$

$$\chi_2(r) = \frac{5\delta^4}{8r^4}\left[\frac{\delta}{r}\sin\frac{2r}{\delta}-2\cos\frac{2r}{\delta}+\frac{\delta}{r}\operatorname{sh}\frac{2r}{\delta}-2\operatorname{ch}\frac{2r}{\delta}\right],$$

$$k_2 = \frac{r_0/\delta}{\left(\sin\frac{r_0}{\delta}\operatorname{ch}\frac{r_0}{\delta}\right)^2+\left(\cos\frac{r_0}{\delta}\operatorname{sh}\frac{r_0}{\delta}\right)^2},$$

$$V_0 = \frac{9}{560}\frac{r_0}{8\pi\rho\nu}\left(\frac{m_0}{r_0^3}\right)^2.$$

Figure 4 shows F_r^0 and F_θ^0 as functions of the dimensionless radius r/r_0 for $\delta/r_0 = 0.1$, 0.5, and 1.0. The open curves are for F_r^0. For $\delta/r_0 = 2.0$ the values of F_r^0 and F_θ^0 are nearly zero and cannot be shown on the scales chosen in Fig. 4. It is clear from Fig. 4 that the flow velocity reaches maximum values for $\delta/r_0 = 0.5$. The flow lines for $\delta/r_0 = 0.5$ are shown in Fig. 5.

We present the solution for the oscillating part of the flow when $\delta \ll r_0$ (problem *a*). The solution of Eq. (4.4) which satisfies the necessary boundary conditions has the form

$$\psi_1(r) = -V_1 r_0 \frac{\delta\delta_1^3}{r r_0^3}\left\{(1+i)\frac{r_0^2}{r^2}\left(\frac{r_0}{\delta_1}-\frac{r_0}{\delta}\right)-\left[(1+i)\frac{r_0}{\delta_1}-1\right]e^{-2(1+i)\frac{r-r_0}{\delta}}+\left[(1+i)\frac{r_0}{\delta}-1\right]e^{-2(1+i)\frac{r-r_0}{\delta_1}}\right\};$$

$$V_1 = \frac{9}{64}\frac{r_0}{8\pi\rho\nu}\left(\frac{m_0}{r_0^3}\right)^2.$$

Hence it is clear that the amplitude of the radial component v_r^1 of the velocity of the oscillating part of the flow increases very rapidly from zero at $r = r_0$ to its maximum $v_{max}^1 = \sqrt{2}\,\frac{\delta\delta_1^2}{r_0^3}\,V_1$ at $r \approx r_0 + \delta$, and then falls off as $(r_0/r)^3$. The v_θ^1 component behaves similarly; it reaches a maximum value twice as large as that of v_r^1 and falls off as $(r_0/r)^4$.

A comparison of v_{max}^1 with the maximum value of the radial component of the velocity of steady motion, $v_{max}^0 = (105/4)\ (\delta^2/r_0^2)V_0$ for $\delta \ll r_0$, shows that their ratio is

$$\frac{v_{max}^1}{v_{max}^0} \sim \frac{\delta_1}{r_0}\frac{\delta_1}{\delta} = \frac{\delta}{r_0}\left(\frac{\delta_1}{\delta}\right)^2 \lll 1;$$

i.e., the velocity of the oscillating flow is very small in comparison with that of the steady flow.

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